ME525 Applied Acoustics Lecture 9, Winter 2022
The acoustic point source (monopole)
The Green's function
Boundary conditions and acoustic doublet

Peter H. Dahl, University of Washington

Continuing with $ka << 1$ and $kL << 1$ limit, monopole source and acoustically compact source

In previous lecture we invoked the $ka << 1$ limit, and with minor rearrangement the pressure is

$$p(r,t) = -i\omega(\rho_0 u_0 4\pi a^2)\frac{e^{ikr}}{4\pi r}e^{-i\omega t} \tag{1}$$

Thus the strength of this acoustic source is defined by the time derivative of *mass flow*, or described another way, it is the *rate of change of mass flow* introduced per unit volume.

After this we defined an *effective source strength* by bundling everything and putting $q = -i\omega(\rho_0 u_0 4\pi a^2)$ giving,

$$p(r,t) = \frac{q}{4\pi r}e^{ikr-i\omega t} \tag{2}$$

where the source is at the center of the coordinate system and pressure is function only of radial coordinate r. So, this source no longer has any length scale a. This length scale has been removed on the assumption that $ka << 1$, and we can replace the sphere of radius a with effective source of strength q.

This defines the concept of concept of a point-like source or *acoustic monopole*. Such a source will generate wave motion in no preferred direction, producing a wave which spreads spherically outward. If there no boundaries, i.e., the medium is infinite in extent, the waveform depend only on the range r from the center of the source, and not depend on the spherical angles α, ϕ as shown in Fig. 1 of Lecture 7.

Furthermore, such a source need not have originally in the form of an exact sphere. The source may instead have some complicated shape (Fig. 1) with characteristic length scale L. We arrive an extraordinarily useful rule: if the characteristic scale L of source is such that $L << \lambda$ where λ is the acoustic wavelength, then the source is *acoustically compact*, and can be viewed as a monopole source. Once the source is deemed acoustically compact the scale L is no longer relevant. The source can be modeling as Eq. (2), where the source strength, q is determined empirically by measurement.

For example, if p_{rms} is measured at range r m from the source, then we can estimate $|q|$ as follows

$$\frac{|q|}{4\pi}\frac{1}{\sqrt{2}}\frac{1}{r} = p_{rms} \tag{3}$$

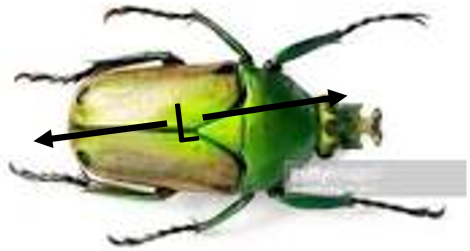

Figure 1: An acoustic source with characteristic scale L for which $kL << 1$ which can be modeled as an *acoustically compact* or source, or monopole.

giving at least a value for $|q|$. Often that is all we are after as the important physics relating to sound propagation is embodied in the factor $\frac{e^{ikr}}{r}$.

Continuing with the Green's function

We next further generalized to find the pressure at a *field point* $\vec{r}$, given a source at an arbitrary *source point* $\vec{r_0}$ that need not be at origin (Fig. 2) as follows:

$$p(\vec{r},t) = \frac{q}{4\pi|\vec{r}-\vec{r_0}|}e^{ik|\vec{r}-\vec{r_0}|-i\omega t} \tag{4}$$

Equation (4) satisfies the inhomogeneous Helmholtz equation, for which the delta function on the RHS represents a point source of strength q at position $\vec{r_0}$ such that

$$(\nabla^2 + k^2)p = -q\delta(\vec{r}-\vec{r_0}) \tag{5}$$

Further compress notation by defining $R = |\vec{r}-\vec{r_0}|$, such that

$$g = \frac{e^{ikR}}{4\pi R} \tag{6}$$

and call g the *free space* Green's function because g satisifies

$$(\nabla^2 + k^2)g = -\delta(\vec{r}-\vec{r_0}) \tag{7}$$

in an *unbounded* medium.

Note the physical dimension of g is $1/L$. As currently constructed, g embodies all the range-dependent and phase properties of a sound field with point source located at $\vec{r_0}$, but to bring a more useful dimension of pressure, g must be multiplied by a calibration constant. The particular form of g in Eq.(6) which concentrated or "impulse-like" in space is known as a *harmonic* Green's function. In this course we use primarily harmonic Green's function solutions, represent-

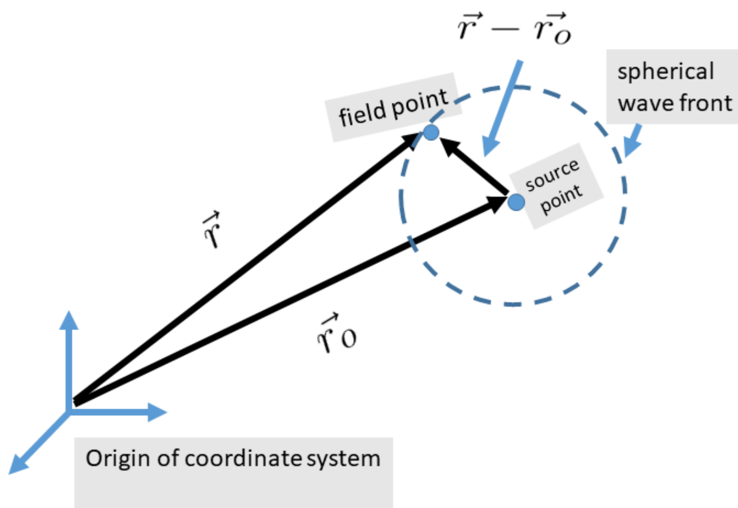

Figure 2: An acoustic source at the source point $\vec{r_0}$ producing the acoustic field at field point $\vec{r}$.

ing a single-frequency, or narrow band condition, and by Fourier superposition we can combine multiple frequencies. A Green's function concentrated in both space and impulsive time is discussed in Pierce (1989), see also Tolstoy (1973).

The Green's function is a model for sound propagation that is proportional to acoustic pressure, differing from pressure only by some multiplicative constant that can complex. The constant may already be known from theory, or determined empirically by measurement. In many applications the constant isn't used as the Green's function usually embodies most if not all the important physics sound propagation.

The expression by way Eq.(5) is common in engineering and physics, where a field quantity, here sound pressure, is governed by a linear partial differential equation with an inhomogeneity at location $\vec{r_0}$ acting as a source term. This pressure field is everywhere smooth and analytic (possesses spatial derivatives that are not infinite), except at the source point it can only be described by a delta function inhomogeneity. This is not unlike a wave created on water, for which the wave field is analytic–except at the point on that surface where the rock splashed (the location of delta function) and produced this wave. In summary: the acoustic field generated by a delta function inhomogeniety is sometimes formally referred to as the *Green's function for the problem.*

Combination of two point sources to satisfy a boundary condition: the acoustic doublet

The Green's function of Eq.(6) is our most basic one which does not need to contend with boundary conditions such as an air-water interface. A simple but realistic boundary condition encountered in acoustics is an acoustic source below an air-water interface. This is called the Lloyd mirror problem and the boundary condition is called "pressure release" meaning the acoustic pressure field must vanish on the boundary. Such boundary, again when viewed from source below, is also referred to as an 'acoustically soft' surface. The 'soft' description arises from the characteristic impedance of two acoustic media involved in the problem: water where ρc is of order 1.5 10^6, and air where ρc is $\sim$ 400. In general, sound impinging on a boundary backed by one medium with characteristic impedance much less than the medium supporting the incident field will also behave approximately as a pressure release boundary, or be 'acoustically soft' .

Note that opposite happens for airborne sound over water; in this case the upper medium is supporting the incident field, and with characteristic impedance of the upper air medium being much less than of the lower water medium the boundary can be considered 'acoustically hard'. Later more general boundary conditions are addressed, as say between soft and hard tissue in medical ultrasound, or between sea water and the seabed, which requires a description of the material properties in terms of the characteristic impedance involved in each boundary material. For now, though, so long as an 'acoustically soft' (or 'acoustically hard') approximate description applies, some very combinations of Green's functions can be applied to satisfy the boundary condition.

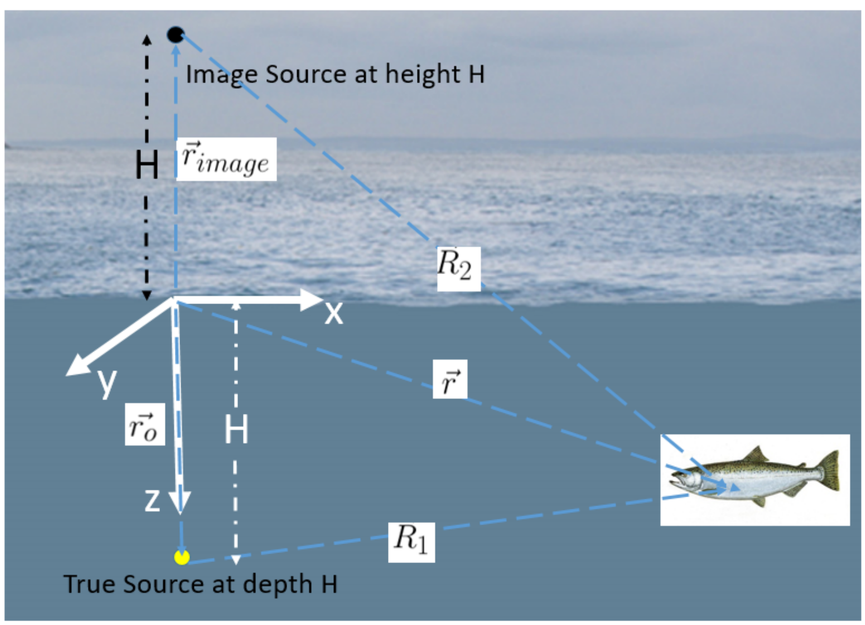

Figure 3: The geometry of the Lloyd Mirror problem showing a true source at depth H and image source at height H.

The pressure release boundary condition is satisfied by combining the true source with an image source (Fig. 3). The image is of opposite sign such that combination of two sources along the boundary equals zero. In this case the Green's function take the following form

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{8}$$

where $R_1 = |\vec{r} - \vec{r_0}|$ and $R_2 = |\vec{r} - \vec{r_{image}}|$. We can set $\vec{r_0} = [0, 0, H]$ and $\vec{r}_{image} = [0, 0, -H]$ in terms of the x, y, z coordinates.

The combination of two free-space Green's functions as in Eq. (8) is known as *doublet*. You should convince yourself that this g equals 0 when evaluated at any x, y with $z = 0$, and remember that g is serving as a surrogate for pressure.

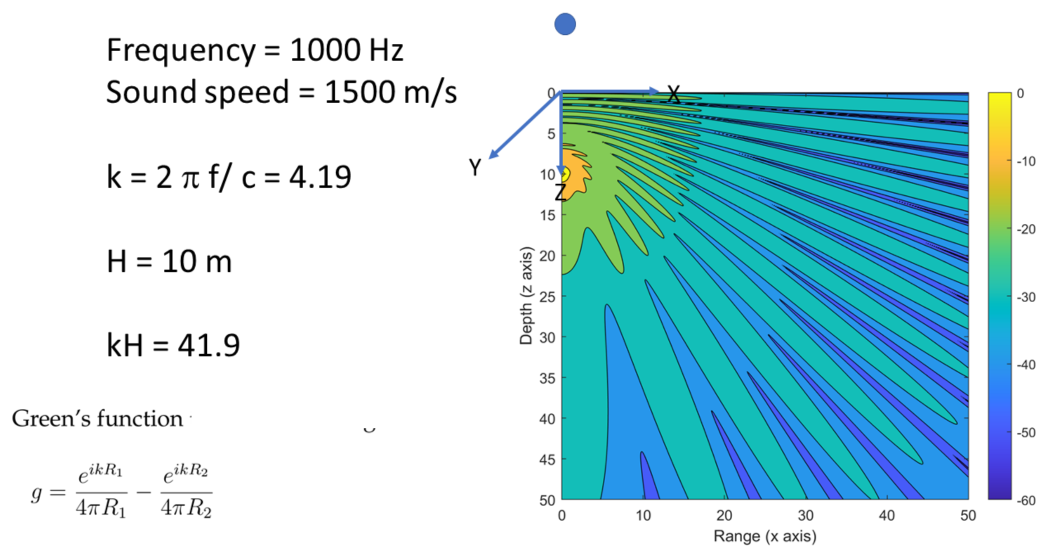

Figure 4: Field in dB with arbitrary reference for acoustic doublet for $H = 10$ and kH = 41.9

Three situations are discussed next for doublet based on differing depths H of the source (Fig. 4-6), and we'll see that a key parameter is kH. In such plots we are interested in how the "strength" of g or pressure, varies with x and z, and so $|g|$ (proportional to pressure amplitude) or $|g|^2$ (proportional to pressure-squared) is plotted. It thus makes sense to continue using decibels, and plot $10 \log_{10} |g|^2$. These three plots display azimuthal symmetry which makes it equally instructive to just plot $20 \log_{10} |g|$ in the x, z plane. Notice what is happening for when the parameter kH is getting smaller.

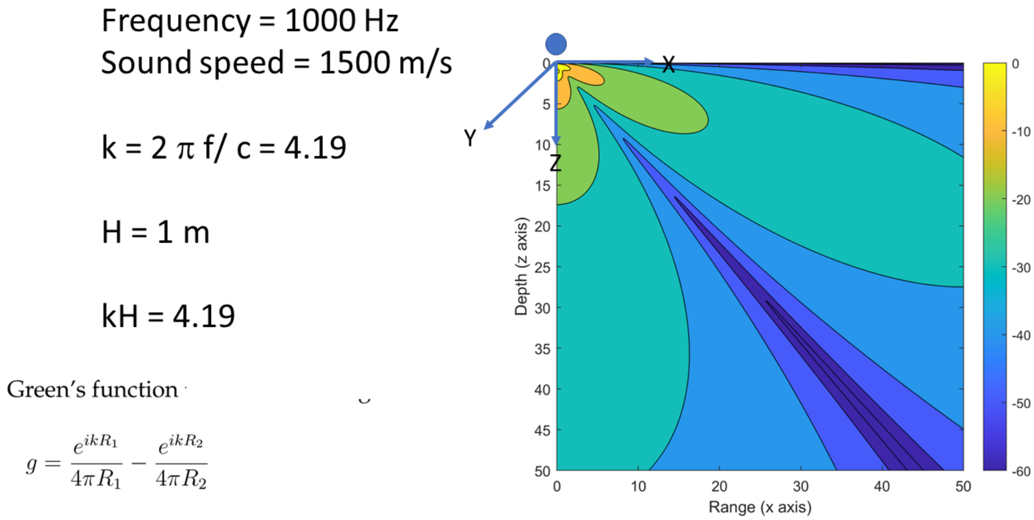

Figure 5: Field in dB with arbitrary reference for acoustic doublet for $H = 1$ and kH = 4.19

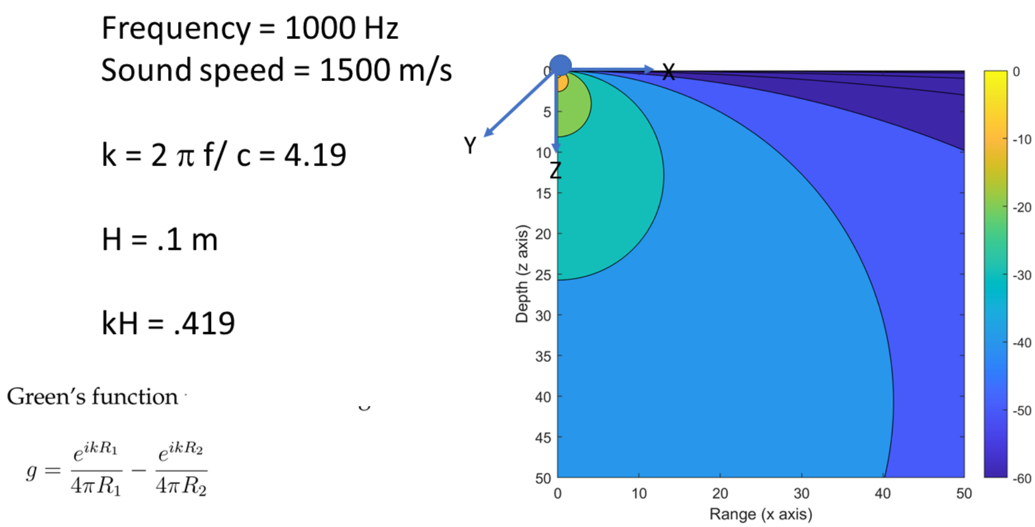

Figure 6: Field in dB with arbitrary reference for acoustic doublet for $H = 0.1$ and kH = 0.419

References

Pierce, A. B, *Acoustics, An Introduction to its Physical Principals and Applications*, (Acoustical Society of America, and American Institute of Physics, 1989)

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L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, (John Wiley & Sons, New York, 1980)

ME525 Applied Acoustics Lecture 10, Winter 2022
the Green's function
acoustic doublet and dipole

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More on combination of two point sources to satisfy a boundary condition, and method of images

It pays to think about boundary conditions, e.g., between soft and hard tissue in medical ultrasound, or between sea water and the seabed, in terms of the characteristic impedance, $\rho_0 c$, of the acoustic medium on each side of the boundary. This does not tell the whole story, e.g., on one side or both sides there may be layers composed of differing sound speeds and densities, but it gives a good starting approximation. But roughly speaking, for sound in a medium with high $\rho_0 c$ impinging on a boundary to another medium with very low $\rho_0 c$, the boundary condition will behave approximately as a 'pressure release' boundary, meaning the acoustic pressure must equal zero along this boundary. In the study of underwater acoustics, this pressure-release boundary conditions is assumed to be exact. You should examine your self the following ratio: $\frac{(\rho c)_{air}}{(\rho c)_{water}}$ using nominal values for each.

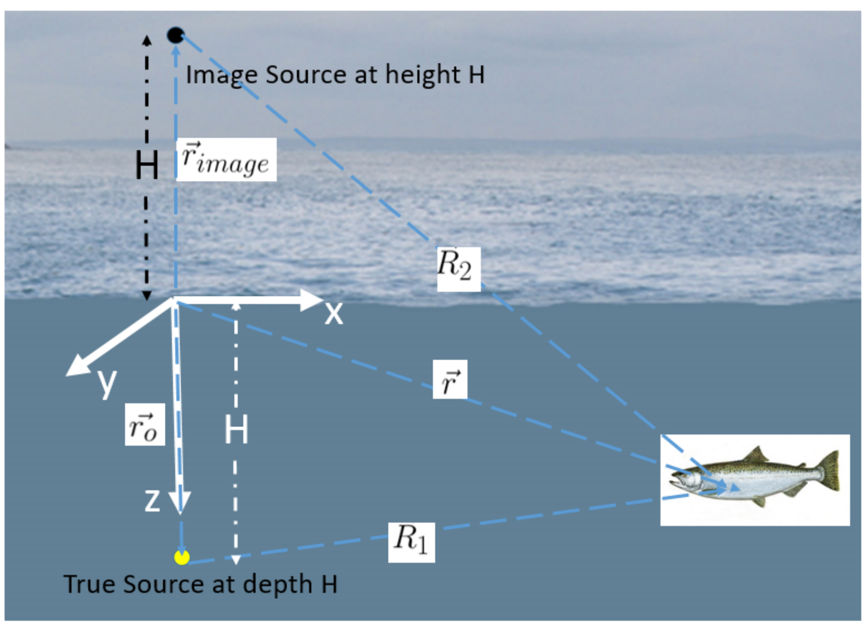

Figure 1: The geometry of the Lloyd Mirror problem showing a true source at depth H and image source at height H.

The pressure release boundary condition is satisfied by combining the true source with an image source (Fig. 1). The image is of opposite sign such that combination of two sources along the boundary equals zero. In this case the Green's function for the Lloyd Mirror problem takes the following form

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{1}$$

where $R_1 = |\vec{r} - \vec{r_0}|$ and $R_2 = |\vec{r} - \vec{r_{image}}|$. We can set $\vec{r_0} = [0, 0, H]$ and $\vec{r}_{image} = [0, 0, -H]$ in terms of the x, y, z coordinates. The method used to find this Green's function is known as the *method of images* (Frisk).

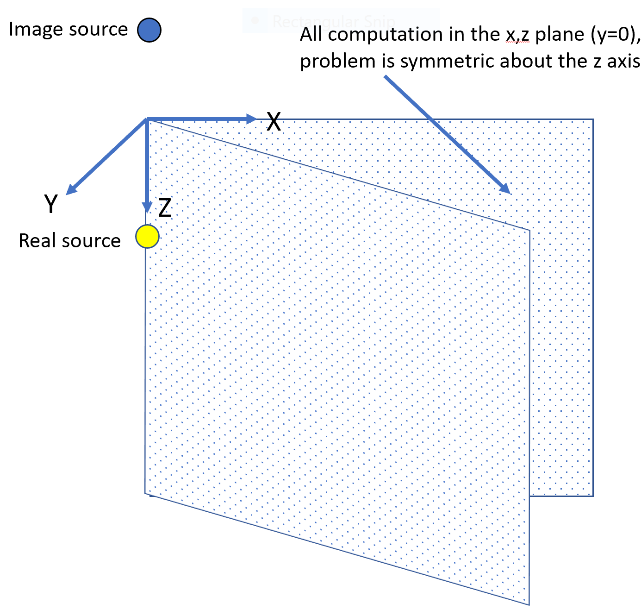

Figure 2: The acoustic field for the geometry in Fig. 1 is symmetric about the z-axis. Thus computing as function of x, z with $y = 0$ is completely sufficient.

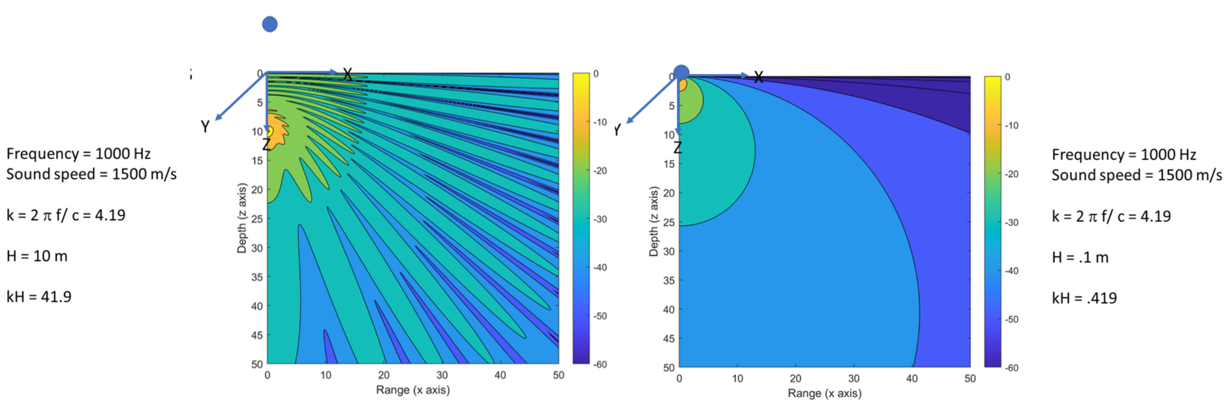

Figure 3: Field in dB with arbitrary reference for two cases with same frequency but differing H as parameterized by $kH >> 1$ and $KH < 1$ acoustic doublet left: $H = 10$ m and kH = 41.9, right: $H = 0.1$ m and kH = 0.419. Similar plots in larger scale are shown in the power point as part of Lecture 9.

Considering the symmetry of the acoustic field about the z axis (Fig. 2), we compute two cases with differing H as parameterized by $kH >> 1$ and $KH < 1$ (Fig. 3). Note that we can also adjust the parameter kH by keeping H the same but adjusting the frequency which changes k. The plots are expressed in terms of contours of $10\log_{10}|g|^2$, or $20\log_{10}|g|$, in the x, z plane, a quantity we might convey to someone else (your co-worker, your research adviser, whomever) in terms of decibels (dB). But the dBs here are obviously not the same as, say, sound pressure level (SPL) in dB reference to 20μPa (air) or 1μPa (underwater). However, considering the fact that g is proportional to acoustic pressure, then there exists some constant-decibel offset to convert results in Fig. 3 to a SPL in dB reference to 1μPa.

We may not be interested in that particular constant-decibel offset, as the more interesting effects are contained in the properties of g as shown here being a function of x, z. Why plot $10\log_{10}|g|^2$ and not just $|g|$ or $|g|^2$? Notice the case $kH >> 1$ with about 14 lobes, or acoustic *beams*, for which $10\log_{10}|g|^2$ varies between about -20 to -30 dB within a beam to about -60 dB outside the beam– let's say a difference of 30 dB or about 1000-fold change in in the value of $|g|^2$. Regions where $|g|^2 \sim -60$ dB are referred to as being in a *null*. We get such a null region because of the interference pattern set up by the (positive) real source and (negative) image source. Though not seen as clearly for case $kH >> 1$, there is very strong null along $z = 0$ boundary, where $|g| = 0$ and any decibel representation would give $-\infty$, a decibel level that is of course not captured in the contour plot. In any case, with such large variation in the field strength it pays to express it in terms of decibel level. (By the way, it is good practice to reserve the word 'level' when talking language involving decibels.)

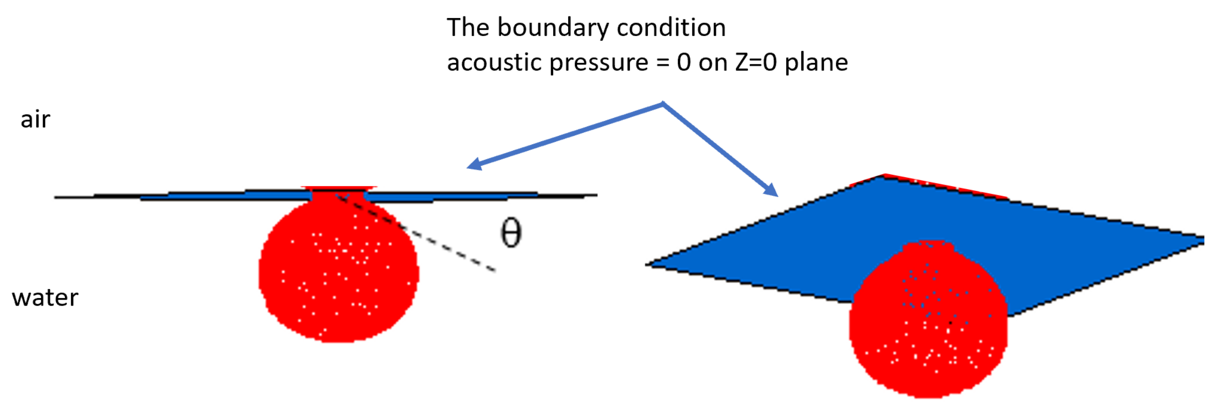

Figure 4: 3D rendition of the very broad beam which results for the acoustic doublet (Fig. 1) when $kH <$.

Turning now to the case $kH < 1$ in Fig. 3, there is also beam and null features, but now the lobes are quite large. For example, take the large -30 dB contour shaded area, and remember this contour swings around 360° about the z axis. It looks something like the rendition in Fig. 4, where one angle with respect to the sea surface, θ, describes everything but dependence on range (an equivalent formulation involves an angle with respect to the z axis).

The acoustic dipole

Now look at the doublet for the case $kH << 1$ (i.e., not too different from Fig. 3 case $kH < 1$) , where $2H$ is the separation between a source and its image which as opposite sign. This is called a *dipole*.

Using the same coordinate system as in Fig. 1, recast R_1 as

$$R_1 = \sqrt{x^2 + y^2 + z^2 - 2zH + H^2} = r\sqrt{1 - 2zH/r^2 + H^2/r^2} \quad (2)$$

then evaluate

$$kR_1 = r\sqrt{k^2 - 2kzH/r^2 + (kH)^2/r^2} \quad (3)$$

In the limit $kH << 1$ the last term is ignored thus $R_1 \approx r(1 - zH/r^2)$, or put $R_1 = r - H\sin\theta$. Similarly we put $R_2 = r + H\sin\theta$. Now, the source $\vec{r_0}$ and image $\vec{r}_{image}$ locations are out of the picture, everything is described by r and θ as in Fig. 5 (the depth coordinate z of the field point included in θ.)

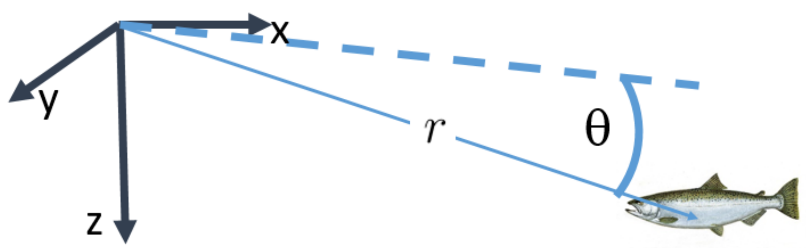

Figure 5: Situation describing a dipole located at center of coordinate system. The acoustic field is function of r and θ.

Now start with the Green's function that satisfies the free-surface boundary condition (discussed previously) composed of source and image of opposite sign (or doublet).

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \quad (4)$$

where $R_1 = |\vec{r} - \vec{r_0}|$ and $R_2 = |\vec{r} - \vec{r_{image}}|$. Now analyze the doublet for the case $kH << 1$, where $2H$ is the separation between a source and its image. This is called a *dipole*.

The approximations for R_1 and R_2, can now be inserted into Eq.(4). In doing so, encounter $e^{\pm ikH\sin\theta}$, which can be approximated as $1 \pm ikH\sin\theta$, consistent with the original $kH << 1$ assumption. This leads to the final result for the dipole Green's function

$$g = 2H\sin\theta\frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}) \quad (5)$$

Observe once again the dimension of this Green's function is $1/L$, as is the case for the two monopoles (source and image) used for the Green's function in Eq.(1). However the dipole now has two terms–one which we be will shown to fade away as $kr >> 1$.

References

Frisk, G. V. *Ocean and Seabed Acoustics* (Prentice Hall, Englewood Cliffs, NJ, 1994)

ME525 Applied Acoustics Lecture 11, Winter 2022
Acoustic Dipole

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The acoustic dipole strength $|f_D|$

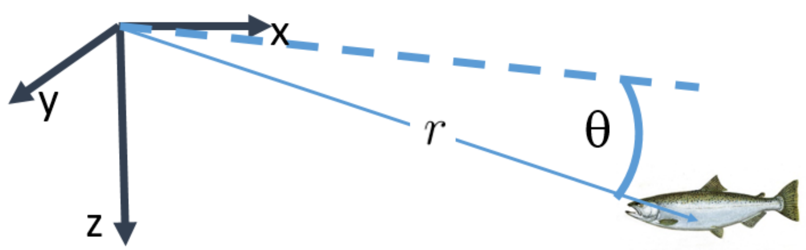

Figure 1: Situation describing a dipole located at center of coordinate system. The acoustic field is function of r and θ.

Let's continue our discussion on the basic dipole Green's function derived in the last lecture.

$$g = 2H\sin\theta\frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}). \tag{1}$$

Recall Eq.(1) originates from the "doublet" consisting of two free-space Green's function sources of opposite sign

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{2}$$

where these sources were separated by H and brought closer together such that $kH << 1$. Next incorporate a new "source strength" q to obtain pressure

$$p(r,t) = (q2H)\sin\theta\frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}) \tag{3}$$

where harmonic time dependence $e^{-i\omega t}$ is assumed.

Apply now the same approach used to establish q for point source monopole, instead here allow $2H$ to shrink to 0 while increasing q to keep $(q2H)$ finite. Identify $|f_D|$ as this new quantity to replace $(q2H)$ which is called the *dipole strength*. Thus, we now more formally express the pressure from the dipole as

$$p(r,t) = |f_D|\sin\theta\frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}). \tag{4}$$

Note the dimension of $|f_D|$: in MKS it must be N/m, or equivalently Pa-m and the dipole unlike the monopole has two terms: one that dominates in the near field, $1/r$, and that dominates the far field ik.

Vector properties of the dipole

The dipole example just presented is one with the axis oriented perpendicular to the boundary (this axis being the line connecting the real source and and image source, or z-axis as in Fig. 1). This is a very appropriate model, for example, for low frequency ship noise where $kH << 1$ with H being the depth of the noise generating mechanism such as the ship propeller, underwater noise caused by rain drops, and by bubbles bursting near the sea surface due to the action of wave breaking.

This dipole example also demonstrates how the boundary between air and water is satisfied upon either placing a dipole on that boundary, or in the case of the doublet, placing a source a distance H below the boundary and a negative image a distance H above the boundary. But more generally we want to place a dipole anywhere in space to represent a source with properties of two closely-space monopole sources of opposite phase (or sign), i.e. there is no requirement to have the dipole source be associated with a boundary. The dipole can be oriented in any direction and this direction will be the *dipole axis*.

To do this, make the strength $|f_D|$ represent the magnitude of a vector $\vec{f}_D$ called the *dipole moment vector*, where $\vec{f}_D$ is aligned with the dipole axis, and by convention points towards the positive side of the dipole. Thus $\vec{f}_D$ points downward in Fig. 1, or in the case just described where the dipole represents a source very close (in the sense of $kH << 1$) to the air-water interface. To use $\vec{f}_D$ in a more general orientation we need to restore a full vector description $\vec{r}$ for the field point, with the angle α (Fig. 2) given by

$$\cos\alpha = \frac{\vec{f}_D \cdot \vec{r}}{|\vec{f}_D||\vec{r}|} \tag{5}$$

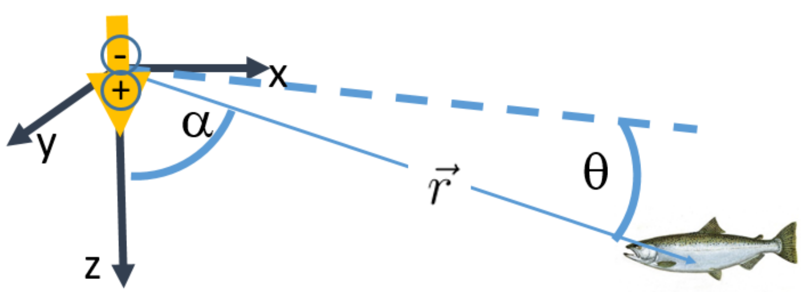

Figure 2: Showing orientation of the dipole moment vector (orange arrow), with positive source ⊕ underwater and negative source ⊖ above. The field point to the fish is now described with vector $\vec{r}$

The dipole moment vector in arbitrary orientation is shown in Fig. 3, and the pressure at the field point (with time dependence $e^{-i\omega t}$) is given by

$$p(FP) = \frac{1}{4\pi}\frac{\vec{f}_D \cdot (\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|}\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}(-ik + \frac{1}{|\vec{r}-\vec{r}_0|}) \tag{6}$$

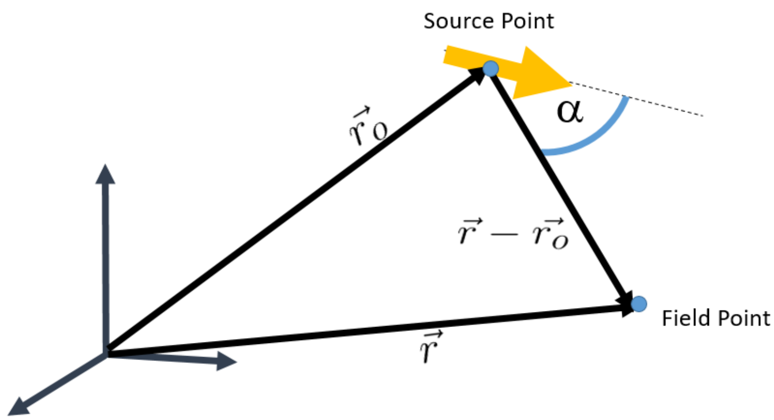

Figure 3: Showing orientation of the dipole moment vector (orange arrow), the dipole source location $\vec{r}_0$ and field point $\vec{r}$ with arbitrary orientation within a coordinate system

Return now to the free space Green's function with harmonic time dependence

$$g = \frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \tag{7}$$

where a pressure at some field point (FP) is

$$p(FP) = qg \tag{8}$$

Note that gradient of g is ∇g is given by (e.g, see Pierce)

$$\nabla g = \frac{1}{4\pi}\frac{(\vec{r}-\vec{r}_0)}{r}(ik - \frac{1}{r})\frac{e^{ikr}}{r} \tag{9}$$

where $r = |\vec{r}-\vec{r}_0|$. Thus evidently Eq. (6) can also be written in the highly compacted form

$$p(FP) = -\vec{f}_D \cdot \nabla g \tag{10}$$

with Eq.(7) and Eq.(9) now representing our two fundamental source types, with monopole related to g and dipole related to ∇g.

Directivity

Dan Russell and colleagues performed a simple, illustrative experiment to demonstrate the concept of directivity from monopoles, dipoles and quadrapoles (a combination of two dipoles). His experiment (Fig. 4) is as follows: four speakers are on turntable and continuous (harmonic or single frequency) sound is broadcast at fixed frequency = 250 Hz. The RMS pressure is measured by a sound level meter (SLM) as the turntable rotates. Speakers (the square boxes) are arranged to be spatially packed together, separated by distance H, say.

The frequency and H are such that $kH << 1$ for this experimental configuration. Therefore, four speakers when broadcasting with the same phase (the black dots) can be considered a single monopole–see (a) top of Fig. 5. Next, take two of the speakers and reverse the wires (polarity) such that the phase is ”negative”, as in (b) top of Fig. 5. This can be considered a single dipole.

The lower part of Fig. 5 shows some results from the Russell *et al.* study. Directively for the monopole is what we intuitively expect: regardless of the position of the turntable, the SLM gives the same result, and this result, rms pressure plotted in terms of dB, forms a circle. In contrast, the dipole directivity exhibits a deep null in the acoustic response as the turntable passes through 90 and 270°. The two halves of the dipole (in this case the two positive and two negative speakers) exhibit an exact phase cancellation and the acoustic pressure resulting from the coherent sum of four sources should vanish. The Russell *et al.* work also measures the acoustic field from a quadrapole (see (d) in upper part of Fig. 5). This will exhibit a kind of clover-leaf directivity pattern.

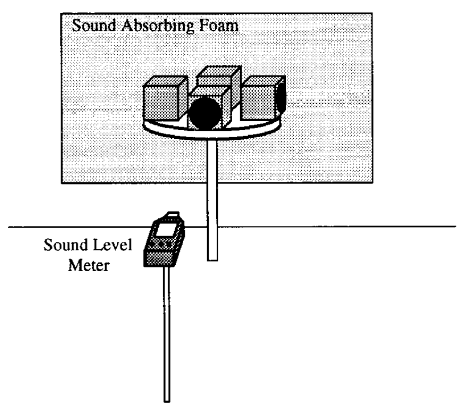

Figure 4: Apparatus for measuring directivity of monopoles, dipoles, and quadrapoles. This is Fig. 5 from Russell *et al.*, ”Monopoles, Dipoles and Quadrapoles: An experiment revisited”, Am. Journal of Physics, 1999.

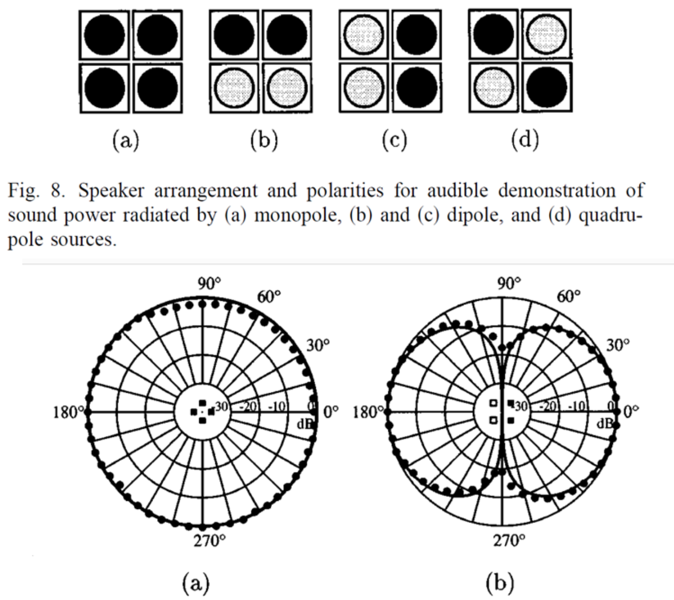

Figure 5: Measurements of directivity pattern for (a) monopole and (b) dipole. This is portion of Fig. 8 from Russell *et al.*, "Monopoles, Dipoles and Quadrapoles: An experiment revisited", Am. Journal of Physics, 1999.

Arrangement of 4 speakers, separated by H and driven by a frequency such that $kH << 1$ amounts to an idealized directivity problem where measurement well predicted by simple theory. More often for realistic noise emission problems, an empirical measurement is required as in the case of directivity of jet noise measured on the ground (Fig. 6).

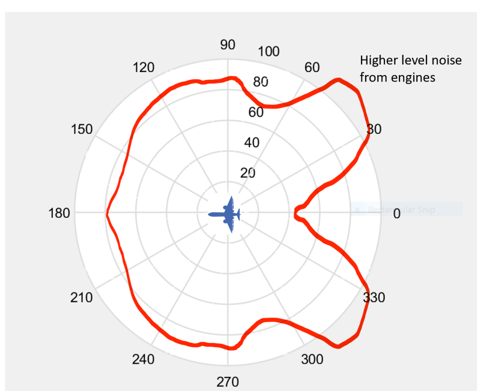

Figure 6: Notional directively of jet noise emission over broad range of frequencies as function of angle.

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ME525 Applied Acoustics Lecture 12, Winter 2022 Combining monopoles and dipoles: The Helmholtz-Kirchhoff (H-K) integral An exact solution to the H-K integral

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Superposition of point sources

Using the free space Green's function representing a source at $\vec{r}_0$

$$g = \frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \tag{1}$$

we can find the total pressure for a superposition of n point sources (Fig. 1) as

$$p(r) = \frac{q_1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|} + \frac{q_2}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}_2|}}{|\vec{r}-\vec{r}_2|} + ... + \frac{q_n}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}_n|}}{|\vec{r}-\vec{r}_n|} \tag{2}$$

allow the source strength for the n source to be q_n, with $e^{-i\omega t}$ time dependence assumed. (Note also that a different frequency ω can be used for each of the n sources.)

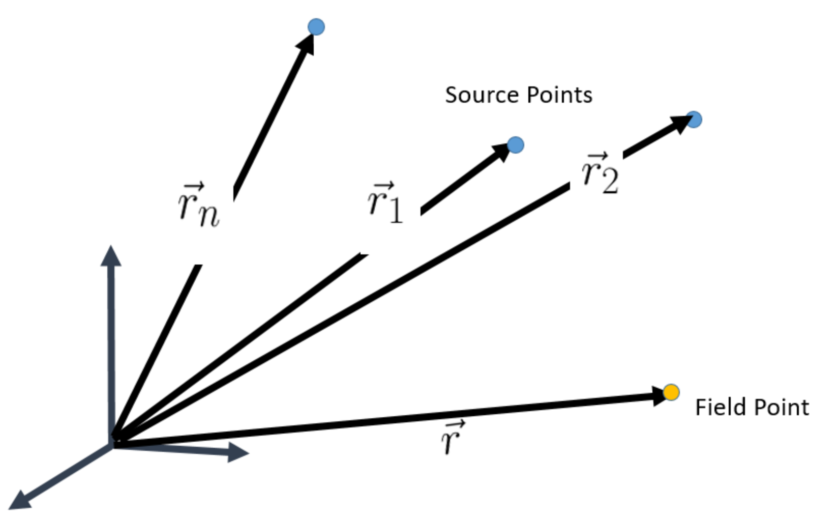

Figure 1: Superposition of n point sources composing the total acoustic pressure at the field point $\vec{r}$

Similarly, there can be continuous distribution of point sources found within a volume V_s–like a sphere fill with marbles each being a point source–with any particular source at $\vec{r}_s$ having a source strength $q(\vec{r}_s)$. In this case the total pressure is the integral over the volume V_s

$$p(r) = \int_{V_s} q(\vec{r}_s) g(\vec{r}, \vec{r}_s) dV_s \tag{3}$$

recalling the notation that $g(\vec{r}, \vec{r}_s)$ is Eq.(1) with $\vec{r}_0 = \vec{r}_s$. As a quick check, if the source distribution consisted of one discrete source, say at position $\vec{r}_0$ inside the volume, then $q(\vec{r}_s) = q\delta(\vec{r}_s - \vec{r}_0)$, and integral over V_s yields $p(r) = qg(\vec{r}, \vec{r}_0)$ in view of the sifting property of the delta function.

A vibratory surface consisting of a surface distribution of elementary sources

Next apply the superposition principle to construct a surface distribution of sources. Figure 2 depicts a vibrating 3D surface of spherical shape generating sound. As a first attempt, model sound generation by a surface distribution of monopole sources with constant velocity amplitude on the surface such that the elemental pressure due to a single patch of surface dS located at $\vec{r}_s$ is

$$dp = -i\omega\rho_0 u_n(\vec{r}_s) g(\vec{r}, \vec{r}_s) dS \tag{4}$$

where $u_n(\vec{r}_s)$ is the normal velocity on the surface at $\vec{r}_s$. Notice that for this problem we have suspended use of source strength q in favor of an explicit value of the normal acceleration $-i\omega u_n(\vec{r}_s)$ times surface area dS. However, do look back onto the discussion in Lecture 8, where the concept of q was introduced and confirm that q has the same dimension as $-i\omega u_n(\vec{r}_s)dS$, i.e., the elemental dS takes on the role of sphere area a^2.

The thick, black line (arrow) in Fig. 2 represents the vector $\vec{r} - \vec{r}_s$ connecting one of the monopole sources related to surface area dS located at $\vec{r}_s$ (or source point) to some arbitrary field point $\vec{r}$. (These $\vec{r}_s$ and $\vec{r}$ vectors are not explicity shown but would be associated with a coordinate system as in Fig. 1.)

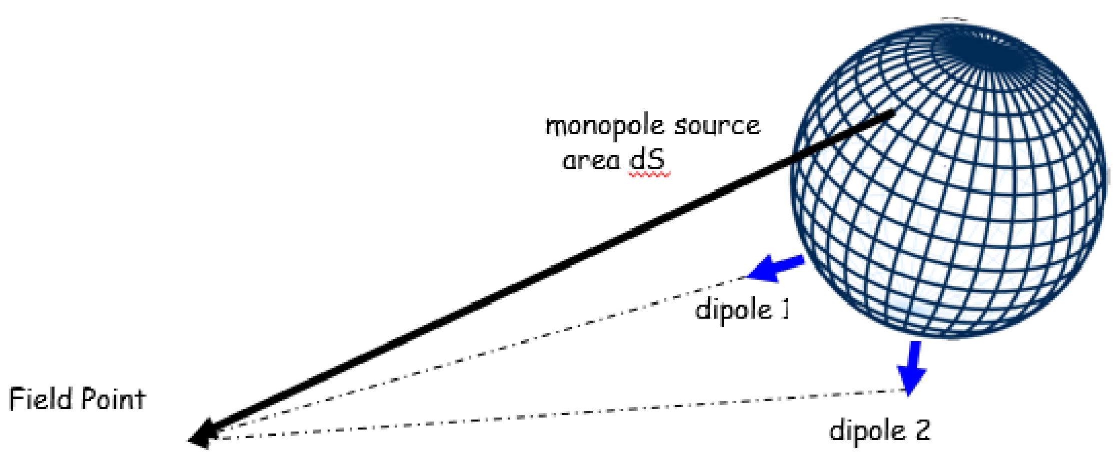

Figure 2: Spherical sound source consisting of surface distribution of elementary sources. Radiation from one monopole elementary source to a field point is shown by solid black line; radiation from two dipole elementary sources is shown by the dotted lines.

It would be really convenient if we could just find the total pressure radiated from this source by integrating over the surface S, or basically summing up all the elemental pressure contributions dp from each dS as follows:

$$p_m(\vec{r}) = \int_S -i\omega\rho_0 u_n(\vec{r}_s) g(\vec{r}, \vec{r}_s) dS \tag{5}$$

where the subscript m ties this pressure to a distribution of monopoles over surface S. This is indeed possible for some geometries which leads to an approach of very practical use, however, for an arbitrary 3D vibrating surface as in Fig. 2, Eq.(5) does not give the complete story.

But for such a non-planer surface of high curvature there will in general will be a need for surface distribution of dipole sources. The two thin, dashed lines sepresent the vectors $\vec{r} - \vec{r}_s$ connecting two different dipole sources. Both dipoles have their axes normal to the surface, but dipole 1 is expected to contribute more to the total pressure at the field point owing to the orientation of the dipole axis with the field point. The problem lies in the fact that when distributed over such a highly curved surface as in Fig. 2, elemental vibrating sources as in monopoles each with contribution dp, cannot assume to be in isolation and operating in free space. This is because other parts of surface S tend to shadow, reflect, and otherwise exert an influence by generating a pressure distribution on the surface of the vibrating body $p(\vec{r}_s)$ that must be accounted for (Fahy 2001, Tempkin 2001).

Such an accounting is made by including a distribution of elemental dipole sources as follows:

$$p_d(\vec{r}) = \int_S p(\vec{r}_s) \frac{\partial g}{\partial n} dS \tag{6}$$

where the subscript d ties this pressure to a distribution of dipoles over surface S. In short, the integral in Eq.(6) propagates the surface pressure distribution $p(\vec{r}_s)$ to the field point $\vec{r}$, where p_m and p_d must add coherently. Thus the total pressure can be written as follows:

$$p(\vec{r}) = \int_S [p(\vec{r}_s) \frac{\partial g}{\partial n} - i\omega\rho_0 u_n(\vec{r}_s) g(\vec{r}, \vec{r}_s)] dS \tag{7}$$

This is known as the *Helmholtz-Kirchhoff integral*.

The Helmholtz-Kirchhoff (H-K) integral poses many challenges to solving. One reason is the first integral is generally more difficult to evaluate than the second. Also, in many instances one may not know the strength of each dipole which is defined by, or linked to, the surface pressure $p(\vec{r}_s)$ at that same position.

For example, a vibrating surface, such as an engine (Fig. 3) might be modeled with constant harmonic velocity amplitude on the engine surface, u_o, associated with engine vibration (i.e, an acceleration amplitude of ωu_o.) Modeling sound radiation from the engine thus involves the H-K integral with $u_n(\vec{r}_s)$ on the engine surface set to u_o. However this does not necessarily specify the surface pressure $p(\vec{r}_s)$ and the H-K integral needs to be recast as an integral equation for $p(\vec{r}_s)$ and

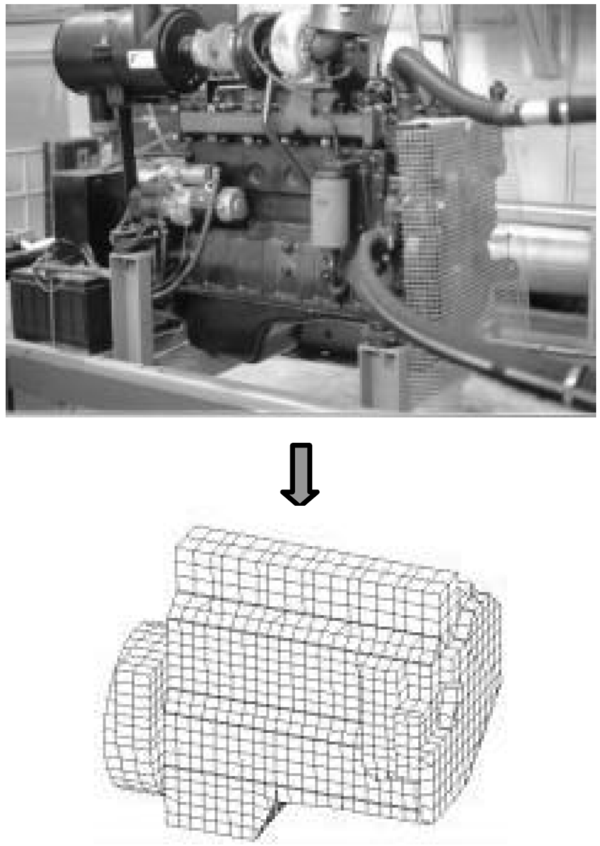

Figure 3: Boundary mesh for the boundary integral method (BEM) analysis for study of noise emission from a diesel engine. This is Fig. 5 from D. W. Herrin *et al.*

generally solved numerically. Often this is a very difficult challenge; some numerical approaches include the boundary element method or BEM.

In ME525 we instead move forward with a more simple, but extremely practical, alternative to summing elemental sources over an area of a radiating surface, known as the *Rayleigh Integral*. However before leaving the H-K integral it is instructive to both drill down a bit more on the differential pressure associated with the dipole contribution, and also study an example of an exact solution for it; this will help to better inform us on the appropriate usage of the Rayleigh integral.

Within the H-K integral the dipole moment vector as part of an elemental source at some position on the surface $\vec{r}_s$ cannot take just any orientation but must align with the outward normal $\vec{n}$ at position $\vec{r}_s$ on the surface. A quick explanation for this is that the dipole strength must come from the pressure at that specific point, $p(\vec{r}_s)$, which can only act normally on the surface, hence the dipole moment vector must also be normal to the surface.

For the surface in Fig. 2, the two dipoles are shown at different locations and dipole moment vectors (blue arrows) are shown aligned with the surface normal at these locations. To obtain the pressure at the field point the dot product of the moment vectors with the corresponding $\vec{r} - \vec{r}_s$ vectors (dotted lines in Fig. 2) is needed, and on this basis it is easy to see that dipole 2 contributes much less than dipole 1 owing to the much smaller dot product.

The differential pressure associated with the dipole contribution is,

$$dp = p(\vec{r}_s)\frac{\partial g}{\partial n}dS. \tag{8}$$

Confirm yourself that Eq.(8) has correct dimension, given that $\frac{\partial g}{\partial n}$ is the spatial derivative of g with respect to an outward normal $\vec{n}$ at position $\vec{r}_s$ on the surface. For the surface dipoles in Fig. 2, identify $[p(\vec{r}_s)dS]\vec{n}$ with the dipole moment vector $\vec{f}_D$ of a simple, isolated dipole first depicted in Fig. 3, Lecture 11. Note further that $\vec{n}\cdot\nabla g = \frac{\partial g}{\partial n}$. It may also be useful in some cases to use

$$\frac{\partial g}{\partial n} = \cos\alpha\frac{\partial g}{\partial R}dS \tag{9}$$

where $R = |\vec{r} - \vec{r}_s|$, with $\frac{\partial g}{\partial R} = \frac{e^{ikR}}{4\pi R}(ik - \frac{1}{R})$. Using this we can again see the interpretation of the angle α between the dipole moment vector and the vector connecting dipole source location with the field point as shown in Lecture 11, Fig. 3.

In summary, think of sound radiation from a general 3D body, as in the sphere shown in Fig. 2 in terms of the H-K integral as a superposition of monopole and dipole sources distributed over the entire, radiating area, of the 3D body. The monopole contribution will have strength $-i\omega\rho_0 u_n(\vec{r}_s)dS$ depending on the particular location $\vec{r}_s$, and each source contribution is propagated to the field point via g which is function of field point $\vec{r}$ and $\vec{r}_s$. The dipole contribution will have strength $p(\vec{r}_s)dS$, propagated to the field point via $\frac{\partial g}{\partial n}$, also a function of $\vec{r}$ and $\vec{r}_s$. By inspection of the basic geometry, e.g., as the case in Fig. 2, one can intuit that many dipole contributions will have little or no effect depending on the orientation of the dipole source with the particular $\vec{r} - \vec{r}_s$ vector.

An exact solution to the Helmholtz-Kirchhoff integral: radiation from a sphere

An exact solution to the Helmholtz-Kirchhoff integral is solve next for a problem we have already solved via a much simpler means: radiation from a sphere of radius a vibrating with constant amplitude over its surface. The problem solved (Lecture 4) involved the specification of a constant velocity on the spherical surface at $r = a$, equal to $u_0e^{-i\omega t}$, where u_0 is a complex amplitude, hence the vibration or acceleration amplitude is $-i\omega u_0$, and we've seen this problem multiple times in homework. The solution is

$$p(r,t) = a\rho_0 c\,\frac{u_0}{r}e^{ik(r-a)}(\frac{ka}{ka+i})e^{-i\omega t} \tag{10}$$

Why do this again? Hopefully the exercise should convince you that the Helmholtz-Kirchhoff integral really works, and one can truly describe sound radiation from a vibrating body as an appropriate sum of monopoles and dipoles. Furthermore, exact, canonical, problems like this can be used as "benchmark" solutions to compare results of more complicated numerical codes.

The H-K integral is Eq.(11) where S is the surface of a sphere of radius a, and the constant u_0 replaces a generally variable normal velocity $u_n(\vec{r}_s)$ on the surface at $\vec{r}_s$

$$p(\vec{r}) = \int_S [p(\vec{r}_s)\frac{\partial g}{\partial n} - i\omega\rho_0 u_0 g(\vec{r}, \vec{r}_s)]dS. \tag{11}$$

There is an unknown pressure over the surface of the sphere $p(\vec{r}_s)$, but symmetry demands that $p(\vec{r}_s)$ not take on any value different from any other value (e.g., as it can for the diesel engine shown in Lecture 11), so it is set to the constant $p(a)$. However this constant pressure value remains unknown and it will have to be determined eventually.

To take further advantage of symmetry, a coordinate system is placed at the center of the sphere (Fig. 4) Now break up the H-K integral, and deal with second integral first, call it I_m

$$I_m = \int_S g(\vec{r}, \vec{r}_s)dS \tag{12}$$

which upon multiplication by $-i\omega\rho_0 u_0$ gives the acoustic field from a uniform distribution of monopole sources over the surface of the sphere.

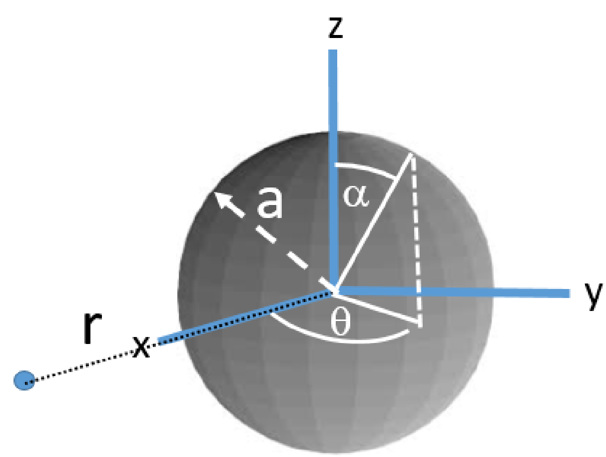

Figure 4: A spherical source of radius a placed at the center of the Cartesian coordinate system (x,y,z). Because of spherical symmetry, one spherical coordinate r describes all variation.

Since the field at point $\vec{r}$ doesn't show angular variation owing to symmetry, a good strategy then is to identify a field point at range r located to be located on one of the Cartesian axes. Set this point to be on the x-axis with $\vec{r} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$, and owing to symmetry this point represents *any* field point.

Now introduce polar angle α and azimuthal angle θ, then all source points $\vec{r_s}$ are described using $\vec{r_s} = \begin{bmatrix} a\sin\alpha\cos\theta \\ a\sin\alpha\sin\theta \\ a\cos\alpha \end{bmatrix}$. Next put $R = |\vec{r} - \vec{r_s}| = \sqrt{r^2 + a^2 - 2ar\cos\alpha}$, and I_m can be reduced to

$$I_m = 2\pi a^2 \int_0^\pi \sin\alpha \frac{e^{ikR}}{R} d\alpha \tag{13}$$

This is readily evaluated, e.g., put $W = \cos\alpha$, $dW = -\sin\alpha d\alpha$, giving

$$I_m = a\sin(ka)\frac{e^{ikr}}{kr}. \tag{14}$$

Unfortunately, $-i\omega\rho_0 u_0 I_m$ is not the complete solution. Interestingly, however, try evaluating both $-i\omega\rho_0 u_0 I_m$ and the known exact solution of Eq.(10) in the $ka << 1$ limit, and find that they are equivalent. Why is that?

Now return to the first integral in Eq. (11), call this I_d because it is the dipole contribution. The unknown pressure on the surface of sphere $p(a)$ can be taken outside the integral, given that we at least know it is a constant, and expressing $\frac{\partial g}{\partial n}$ as $\nabla g \cdot \vec{n}$ then

$$I_d = \int_S \nabla g \cdot \vec{n} dS \tag{15}$$

where $p(a)I_d$ gives the acoustic field from a uniform distribution of dipole sources over the surface of the sphere.

The divergence theorem is used to convert the surface integral over the surface of the sphere into a volume integral over the volume V of the sphere, of $\nabla^2 g$. But now recall that

$$\nabla^2 g + k^2 g = -\delta(|\vec{r} - \vec{r_s}|) \tag{16}$$

Now equate $\nabla^2 g$ to $-k^2 g - \delta(|\vec{r} - \vec{r_s}|)$ and instead integrate these terms over the volume V.

But there are no receiver points, or field points, *inside* the sphere, since we are solving the problem for field points $\vec{r}$ *outside* of the sphere, or $r > a$. Thus, the argument $|\vec{r} - \vec{r_s}|$ *never* becomes 0 inside the volume V and by the definition of the delta function, the volume integral of $\delta(|\vec{r} - \vec{r_s}|)$ will be 0. Thus $I_d = -k^2 \int_V g dV$. Apart from the k^2 factor, I_d describes a volume V (of the sphere) now filled with point monopole sources, at source points $\vec{r_s}$ within the sphere, as result of applying the divergence theorem.

To proceed put $R = |\vec{r} - \vec{r_s}| = \sqrt{r^2 + \rho^2 - 2\rho r\cos\alpha}$, where ρ varies from 0 to a, and I_d becomes

$$I_d = \frac{1}{2}\int_0^a \rho^2 d\rho \int_0^\pi \sin\alpha \frac{e^{ikR}}{R} d\alpha. \tag{17}$$

The integral over α can is more easily solved by putting $w = \cos\alpha$, $d\alpha = \frac{-dw}{\sin\alpha}$ after which you should arrive (maybe try it yourself?) at

$$I_d = (ka\cos ka - \sin ka)\frac{e^{ikr}}{kr} \tag{18}$$

Now, observe that evaluating Eq.(11) at a field point on the surface of the $\vec{r} = a$ (but not inside the sphere), implies

$$p(a) = p(a)I_d(a) - i\omega\rho_0 u_0 I_m(a) \tag{19}$$

and find the unknown pressure $p(a)$ equal to

$$p(a) = \frac{-i\omega\rho_0 u_0 I_m(a)}{1 - I_d(a)} \tag{20}$$

with complete solution

$$p(r) = p(a)I_d(r) - i\omega\rho_0 u_0 I_m(r) \tag{21}$$

the first term being the dipole contribution and the second being the monopole contribution.

Everything in Eq. (21) is represented by the simple formula given in Eq.(10), for the problem we already solved. With more algebraic manipulation they equate precisely. Instead its more interesting to keep it as it is, broken out into its dipole and monopole terms; plot them out and see if the coherent sum matches Eq. (10).

This is done (Fig. 5) for a sphere of radius $a = 0.05$ m, for which there is a uniform velocity amplitude on the surface of the sphere of $u_0 = 0.001$ m/s. The velocity has harmonic dependence $e^{-i\omega t}$, for which the frequency is stepped through from 10 Hz to 20000 Hz in 10 Hz steps. This establishes a range of ka (using c = 330 m/s), which goes from values $<< 1$ to about 20. The mean-square pressure is plotted at range 1 m for the monopole term (red line), dipole term (blue line), their sum (black, dashed line), and finally the green line of Eq. (1).

It should be clear that Eq.(21) works, and we have successfully modeled acoustic emission from a spherical source in terms of a surface distribution of dipoles of strength $p(a)$, and a surface distribution of monopoles of strength $-i\omega\rho_0 u_0$. It's interesting to see that at some particular values of ka either just the monopole or dipole term approximates the exact solution in Eq.(10) reasonably well, as in $ka \sim 9.3$ for the monopole and $ka \sim 11$ for the dipole, with this behavior being periodic ka. Additionally for ka less than about 0.2 the complete solution is well satisfied by the just the monopole term.

We are done with the H-K integral in terms of ME525 involvement. More complicated geometries with curvature but without the kind symmetry we see in this problem, must be solved numerically such as the boundary element method (BEM).

However, if the surface is relatively flat one can imagine that each dipole on the surface, with

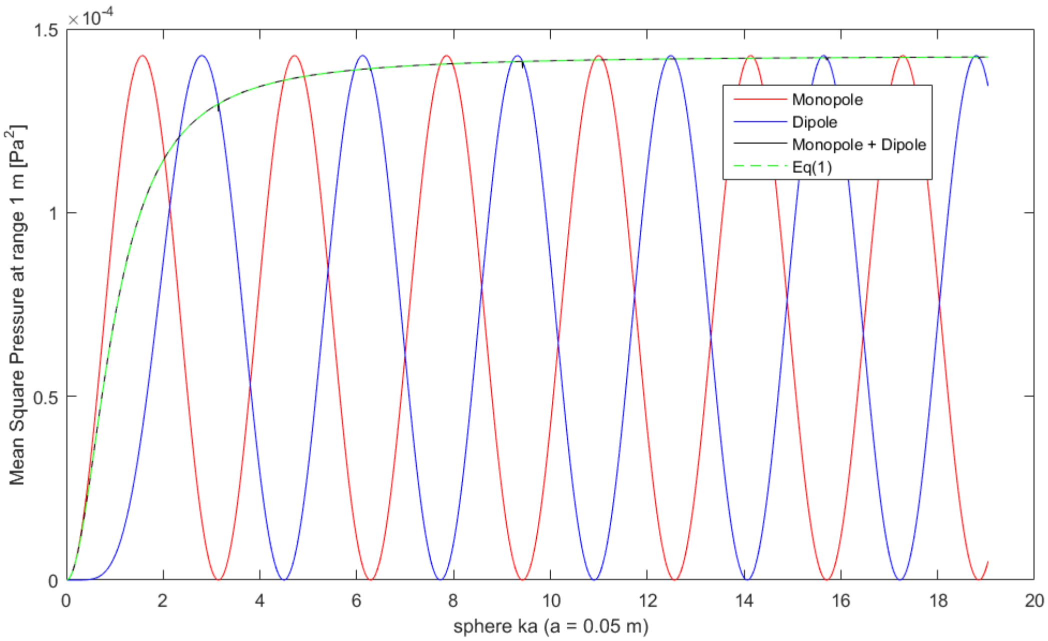

Figure 5: Mean square pressure associated with sound radiation from a sphere of radius $a = 0.05$ m, vibrating with uniform velocity amplitude on the surface of the sphere $u_0 = 0.001$ m/s. Results are plotted as function of ka which is varied by increasing the frequency from 10 Hz to 20000 Hz. Various model representations for the mean-square pressure are shown as discussed in the text.

dipole axis aligned with the local surface normal, will pretty much cancel itself out. Thus the I_d part of the H-K integral can be neglected, in favor of doing the computationally simpler I_m integral. This is the basis behind the Rayleigh integral discussed next and which you will have an opportunity to work with.

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